

Constraining the vertical structure of the Milky Way rotation by microlensing in a finite-width global disk model

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ABSTRACT

In this paper we model the vertical structure of mass distribution of the Milky Way galaxy in the framework of a finite-width global disk model. Assuming the Galactic rotation curve only, we test inside the solar orbit the predictions of the model for two measurable and unrelated to each other processes: the gravitational microlensing that allows to fix the disk width-scale by the best fit to measurements, and the vertical gradient of rotation modelled in the quasi-circular orbits approximation. The former is sensitive to the gravitating mass in compact objects and the latter is sensitive to all kinds of gravitating matter. The analysis points to a small width-scale of the considered disks and, at most, insignificant contribution of non-baryonic dark matter in the solar circle. The predicted high vertical gradient values in the rotation are consistent with the gradient measurements.

Key words. gravitational lensing: micro - Galaxy: disk - Galaxy: kinematics and dynamics - Galaxy: structure - dark matter

1. Introduction

The Milky Way (MW) is an example of a galaxy with high vertical gradients of rotation measured at low altitudes above the mid-plane (Levine et al., 2008). It is interesting to see the effect of the disk thickness on the gradient value prediction and to compare it with the analogous prediction in an infinitesimally thin global disk model (Jałocha et al., 2010).

The gravitational microlensing phenomenon provides another constraint on the mass distribution, independent of the vertical gradient structure. In particular, the amount of mass seen through the gravitational microlensing measurements inside the solar orbit, was shown to be consistent with the dynamical mass ascertained from the Galactic rotation and reduced by the gas contribution undetectable by the microlensing (Sikora et al., 2012).

In the present paper we use microlensing measurements to constrain the disk width and we compare the resulting vertical gradient predictions with the gradient measurements. In our several previous studies, e.g., Jałocha et al. (2010, 2011), we modelled spiral galaxies in the approximation of an infinitesimally thin disk. In this framework, we could easily obtain high values of the vertical gradient of rotation, in accord with gradient measurements. However, the model does not account for the vertical structure in the mass distribution in the direct neighborhood of the mid-plane (the $z = 0$ vicinity). This structure can be modelled by considering a finite-width disk with an assumed vertical profile of mass density. A preliminary step towards this we presented in Sikora et al. (2012), where a column mass density of a disk with the exponential vertical profile was identified with the surface density of an infinitesimally thin disk. But this was a simple de-projection which did not take into account the redistribution of mass required to preserve the shape of rotation curve in the Galactic center vicinity.

In the next approximation considered in this paper, in Sect. 2 and Sect. 3, we find by iterations the volume mass densities of two example finite-width disks. These disks exactly account for the MW's tabulated rotation curve and have their width-scales constrained by the gravitational microlensing measurements. In Sect. 4 we compare the predictions of the two models with the measurements of the vertical gradient of rotation.

2. A finite-width disk model

We assume the following factorized form of a volume mass density $\rho(r, z) = \sigma(r)f(z)$, with factors normalized so that $\sigma(r)$ is the column mass density and the vertical profile $f(z)$ is integrable to unity: $\int_{-\infty}^{\infty} f(z)dz = 1$. In this paper we consider the exponential vertical profile (for its simplicity) and the “mexican hat” profile (for its being frequently used in star count models motivated by Spitzer (1942) analysis):

$$f(z) = \frac{1}{2h'} \exp\left(-\frac{|z|}{h'}\right), \quad f(z) = \frac{1}{2h} \operatorname{sech}^2\left(\frac{z}{h}\right). \quad (1)$$

The width-scale parameters h' and h can be related one to another by equating their effective disk thicknesses: $2h'$ and $\approx 1.49h$, respectively, defined by the “1/e criterion”.¹

The gravitational potential at a point $\mathbf{X} = [R, 0, Z]$ associated with a mass element $dm = \rho(r, z)rdrd\phi dz$ located at another point $\mathbf{Y} = [r \cos \phi, r \sin \phi, z]$, is $\frac{Gdm}{|\mathbf{Y}-\mathbf{X}|}$, where $|\mathbf{Y}-\mathbf{X}| = \sqrt{R^2 + (z-Z)^2 + r^2 - 2rR \cos \phi}$. On substituting $\phi =$

¹ The “1/e criterion” we formulate as follows: the effective width-scale h' of a vertical profile $f(z)$ integrable to unity is defined by comparing its mass inside a shell $|z| < h'$ with that of an exponential one with a scale-width h' : $\int_{-h'}^{h'} f(z)dz \equiv \int_{-h'}^{h'} \frac{e^{-|z|/h'}}{2h'} dz = 1 - \frac{1}{e}$. In particular, for the “mexican hat” profile we obtain $\tanh(h/h') = 1 - \frac{1}{e}$ or $2h' \approx 1.49h$.

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$2\alpha - \pi$, using the symmetry $\phi \rightarrow -\phi$, integrating over $\alpha \in (0, \pi/2)$ and using the definition of the elliptic integral of the first kind K (Gradshtein et al., 2007), one arrives at an expression for the total potential $\Psi(R, Z)$ at X : $\Psi(R, Z) = -4G \int_0^\infty dr \int_{-\infty}^\infty dz \frac{r K\left(\sqrt{\frac{4rR}{(R+r)^2 + (z-Z)^2}}\right) \rho(r, z)}{\sqrt{(R+r)^2 + (z-Z)^2}}$. With the help of the identity $K'(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}$ (Gradshtein et al., 2007), it can be shown that $R \partial_R \Psi(R, Z) = 2G \int_0^\infty dr \int_{-\infty}^\infty dz \rho(r, z) J(r, R, z - Z)$. Function J is an integration kernel that will frequently appear later:

$$J(r, R, \zeta) = \frac{r \cdot \left(K(\kappa) - \frac{r^2 - R^2 + \zeta^2}{(r-R)^2 + \zeta^2} E(\kappa) \right)}{\sqrt{(R+r)^2 + \zeta^2}}, \quad \kappa = \sqrt{\frac{4rR}{(R+r)^2 + \zeta^2}}.$$

The expression for $R \partial_R \Psi(R, Z)$ can be used to compute the rotation velocity $v_\phi(R, Z)$ in the quasi-circular orbits approximation studied in Jałocha et al. (2010). In this approximation, $v_\phi^2(R, Z) = R \partial_R \Psi$, and the resulting vertical gradient of rotation is $\partial_Z v_\phi(R, Z)$. The latter quantity would involve differentiation of the kernel J under the integration sign. However, since $\rho(r, z)$ is of the form $\sigma(r)f(z)$, with $f(z)$ being known in an analytic form, falling off fast enough as $|z| \rightarrow \infty$, and satisfying the reflection symmetry $f(z) = f(-z)$, it is more convenient to integrate by parts. Noting that $\partial_z J = -\partial_Z J$, we are led to the following expressions for the circular velocity and its vertical gradient (by the assumed reflection symmetry of f , the integration has been restricted to $z > 0$):

$$v_\phi^2(R, Z) = 2G \int_0^\infty dr \int_0^\infty dz \rho(r, z) \cdot [J(r, R, z - Z) + J(r, R, z + Z)], \quad (2)$$

$$\partial_Z v_\phi(R, Z) = \frac{G}{v_\phi(R, Z)} \int_0^\infty dr \int_0^\infty dz \partial_z \rho(r, z) \cdot [J(r, R, z - Z) - J(r, R, z + Z)]. \quad (3)$$

(The derivatives of J have been eliminated from Eq. 3 by means of an integration by parts.) It should be stressed that these two expressions are valid in the quasi-circular orbits approximation only.

The above integral expressions are particularly suited for the exponential vertical profile, in which case they reduce to $v_\phi^2(R, Z) = I_- + I_+$ and $\partial_Z v_\phi(R, Z) = \frac{I_+ - I_-}{2h \sqrt{I_- + I_+}}$, where I_- and I_+ are appropriate integrals involving $J(r, R, z - Z)$ and $J(r, R, z + Z)$, respectively.

For a more explicit derivation of the above results, the reader is referred to the Appendix.

2.1. Determining $\rho(r, z)$ from the rotation curve by iterations

We use a smoothed-out MW rotation curve from Sofue et al. (1999), which adopts Galactic constants $R_0 = 8$ kpc and $\Omega_0 = 200 \frac{\text{km}}{\text{s}}$. Its inner part, inside solar circle, which is of interest in our paper, is relatively well determined and was obtained by simple averaging of various CO and HI tangent velocity data. The uncertainty lies mostly in the velocity parameter of the standard of rest at the Sun position, Ω_0 , and the radius of solar orbit R_0 . For larger radii, outside the region of our interest, the rotation curve is less certain and even model dependent. Furthermore, as follows from the analysis in Bratek et al. (2008), the uncertainty

in the external rotation curve implies some uncertainty in the internal mass determination due to a backward-interaction characteristic of flattened mass distributions. However, in Sikora et al. (2012) we found this influence to be marginal for the purpose of the present study. More important is to reduce the uncontrolled numerical errors arising due to the presence of singular kernels in integrals Eq. 2 and Eq. 3. To achieve this, we apply a cubic spline interpolation to the rotation points. This way we obtain a continuously differentiable interpolating rotation curve (see Fig. 1).

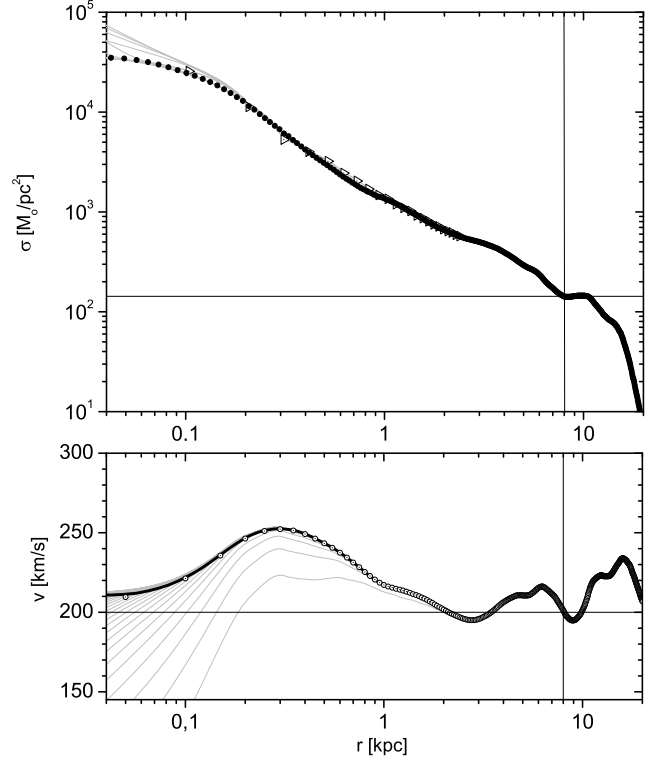


Fig. 1: Column mass densities and the corresponding rotation curves of finite-width disks with a “mexican hat” vertical profile ($h = 117$ pc), shown at several steps of the iteration method of Sect. 2.1, starting from the surface density of an infinitesimally thin disk.

Top panel: [THIN GRAY LINES] – column mass densities at various iteration steps, [TRIANGLES] – the surface mass density of the infinitesimally thin disk model (the starting point of the iteration), [SOLID CIRCLES] – the MW column mass density in the final iteration step (it gives the interpolating rotation curve in the bottom panel).

Bottom panel: [OPEN CIRCLES] – MW rotation curve points from Sofue et al. (1999), [THIN GRAY LINES] – rotation curves at various iteration steps, and [THICK LINE] – rotation curve in the final iteration step (it strictly overlaps with a spline interpolation of the MW rotation curve points).

A primary approximation to the column mass density, $\sigma_0(R)$, we obtain from the interpolating rotation curve in the infinitesimally thin global disk model, by iterations akin to those described in Jałocha et al. (2008), with the help of the following integral transforms which are mutual inverses of each other (Sikora et al., 2012)

$$\sigma_0(R) = \frac{1}{2\pi^2 G} \int_0^\infty \left[\frac{K[k(\chi)]}{1+\chi} - \frac{E[k(\chi)]}{1-\chi} \right] \frac{v^2(R\chi)}{R\chi} d\chi. \quad (4)$$

and

$$\frac{v^2(R)}{R} = 2G \int_0^\infty \left[\frac{K[k(\chi)]}{1+\chi} + \frac{E[k(\chi)]}{1-\chi} \right] \sigma_0(R\chi) \chi d\chi, \quad (5)$$

with $k(\chi) = \frac{2\sqrt{\chi}}{1+\chi}$ in both cases. The transforms relate the rotation law in the Galactic mid-plane to the surface mass density in that plane. Here, R is the radial variable in the disk plane, K and E are complete elliptic integrals of the first and second kind as defined in (Gradshtein et al., 2007), and χ is a dimensionless integration variable. The surface mass density $\sigma_0(r)$ obtained this way gives a primary approximation $\rho_0(r, z) = \sigma_0(r)f(z)$ to the volume mass density, which should be close to the mass density in a finite-width disk (the results of Sikora et al. (2012) were based on this approximation). But evaluating integral Eq. 2 with $\rho_0(r, z)$ substituted for $\rho(r, z)$, gives a $v_{\phi,0}^2(R, 0)$ different from $v_c^2(R)$ by a small amount $\Delta v_{\phi,0}^2(R)$ measuring the discrepancy between the predicted and the observed rotation. Now, by inserting the $\Delta v_{\phi,0}^2(R)$ in place of v^2 in Eq. 4, we obtain a correction $\Delta\sigma_0(R)$ to the column mass density. Hence, the surface mass density in the next approximation is $\sigma_1(R) = \sigma_0(R) + \lambda \cdot \Delta\sigma_0(R)$ and gives rise to a corrected volume mass density $\rho_1(r, z) = \sigma_1(r)f(z)$, which in turn, from Eq. 2, gives the corresponding corrected rotation curve $v_{\phi,1}(R, 0)$. Next, we shift all indices by +1 and repeat the previous step. This correction process can be continued recursively until it converges to the desired density profile $\rho(r, z)$ for which the discrepancy $\Delta v_{\phi,j}^2$ becomes negligible. The resulting recursion sequence of volume densities $\rho_j(R, Z) = \rho_{j-1}(R, Z) + \lambda \cdot f(Z)\Delta\sigma_{j-1}(R)$ is quickly converging to a limit $\rho(R, Z) = \lim_{j \rightarrow \infty} \rho_j(R, Z)$ which, on substituting to Eq. 2, yields a curve $v_{\phi}(R, 0)$ that nicely overlaps with the interpolating rotation curve. The iteration process is illustrated in Fig. 1.

In Sect. 3 we use this method of finding column mass densities $\sigma(R)$ corresponding to various vertical profiles $f(z)$. They all give rise to rotation laws that overlap in the mid-plane with the MW rotation curve.

3. The gravitational microlensing

The phenomenon of the gravitational light deflection in point mass fields, called the gravitational microlensing, may be used to tracing mass distribution. This can be achieved by inferring the amount of mass in compact objects scattered along various lines of sight joining the observer and remote sources of light distributed in the vicinity of the Galactic center. For it is not directly linked to the Galaxy dynamics, the gravitational microlensing provides an independent tool useful in testing Galactic models.

The original idea comes from Paczynski (1986), who suggested that the gravitational lensing can be used to answer the question of whether or not the spheroidal component of the Galaxy could be dominated by massive compact objects. Observations towards Great and Small Magellanic Clouds, e.g., Alcock et al. (2000); Wyrzykowski et al. (2011), indicate that the contribution from such objects cannot be dominating in the standard three-component galactic model. On the other hand, it turns out that the microlensing method can be used also to estimate the distribution of compact objects in the direction towards the Galactic center. Such measurements can help to improve models of the galactic interior, see e.g., Bissantz et al. (2004). It should be pointed out that in most of these models, the dark matter component becomes necessary only beyond some particular distance from the Galactic center. For example, in the model by Bissantz & Gerhard (2002), the dark matter halo is irrelevant for distances smaller than 5 kpc from the center.

² The free parameter λ , can be used to control the rate of convergence of the iteration process. We used $\lambda = \frac{2}{3}$ instead of 1.

The aim of the present work is to examine in the context of the microlensing observations, a galactic model in which the dynamics is dominated by baryonic matter distributed in direct neighborhood of the Galactic mid-plane.

3.1. The optical depth

The most important quantity to be determined in the microlensing method is the optical depth τ . It is defined as the probability of finding a compact object (a lens) on the line of sight between the observer and the source of light, when a lens is located within its Einstein radius

$$R_E = \sqrt{\frac{4GM}{c^2} \frac{D_L(D_S - D_L)}{D_S}}$$

on a plane perpendicular to the line of sight. Here, M denotes the lens mass, D_S is the distance from the observer to the source, and D_L is the distance between the observer and the lens. In this particular configuration, a double image of the source is produced each time a lens passes between the source and the observer. Although the two images cannot be usually resolved due to very small deflection angles, their appearance can still be detected indirectly by measuring the associated image magnification. A microlensing event of this kind is agreed to have been occurred when the magnification exceeds a threshold value of $\mu = 1.34$. Because the probability of microlensing events is very low (of the order of 10^{-6}), a great number of sources (of about a million) must be monitored during a period of few years so as to the optical depth could be determined. A detailed discussion of such observations and their theoretical description can be found in Moniez (2010).

The principle result of the microlensing theory is the following integral that relates the mass density of compact objects $\rho(\mathbf{x})$ and the optical depth τ :

$$\tau = \frac{4\pi G}{c^2} \int_0^{D_S} \frac{D_L(D_S - D_L)}{D_S} \rho(\mathbf{x}(D_L)) dD_L. \quad (6)$$

Although it is well known, we re-derived this formula for completeness in our previous paper (Sikora et al., 2012). The integration in Eq. 6 is carried out along a given line of sight, between the observer's position $\mathbf{x}_0 = [R_0, 0, 0]$ and the source of light located at $\mathbf{x}_\infty = \mathbf{x}_0 + (1 + \chi)\mathbf{R}_0 [-\cos b \cos l, -\cos b \sin l, \sin b]$. The angle l is the galactic longitude, b is the galactic latitude and χ is a dimensionless distance parameter, such that $D_S = R_0(1 + \chi)$. Using the following parameterization of the line of sight $\mathbf{x}(s) = \mathbf{x}_0 + s(\mathbf{x}_\infty - \mathbf{x}_0)$, equation Eq. 6 can be rewritten such that τ becomes an explicit function of the source's position (l, b, χ) (Sikora et al., 2012):

$$\tau(l, b, \chi) = \frac{4\pi G R_0^2}{c^2} \int_0^1 (1 + \chi)^2 s(1 - s) \rho(r(s), z(s)) ds, \quad (7)$$

where $r(s) = R_0 \sqrt{1 + s(1 + \chi) \cos b [s(1 + \chi) \cos b - 2 \cos l]}$ and $z(s) = R_0 s(1 + \chi) \sin b$. Given a density distribution $\rho(r, z)$, the above formula enables one to calculate the corresponding optical depth.

3.2. The data

The model optical depth, calculated with the help of expression Eq. 7, must be compared with the observations. For that purpose, we use data collected by several leading collaborations, in particular: MACHO (Popowski et al., 2005), EROS

(Hamadache et al., 2006), OGLE (Sumi et al., 2006), and MOA (Sumi & et al, 2003). These data were collected and analyzed in detail in a review (Moniez, 2010).

Later, we restrict our analysis to the bright stars subsample. This is a commonly used strategy in minimizing blending processes that affect the optical depth results. A discussion of the blending effect can be found in Alard (1997) and Smith et al. (2007). The data we use is represented as a function of latitude $\tau(b)$, and for each b the optical depth has been averaged over the longitudinal angle in the interval $l \in (-5^\circ, 5^\circ)$. The resulting function $\tau(b)$ allows us to study the vertical structure of the Galaxy.

3.3. Previous results in view of the present study

In our previous paper (Sikora et al., 2012), we used a surface mass density $\sigma(r)$, which accounts for the Galactic rotation curve in the infinitesimally thin disk model, to obtain a volume mass density $\rho(r, z)$ corresponding to $\sigma(r)$, assuming the standard exponential vertical profile $\rho(r, z) = \rho(r, 0) e^{-|z|/h'}$. Following the EROS collaboration, e.g., Derue et al. (2001), we set the value of the width-scale parameter to be $h' = 325$ pc. With such obtained $\rho(r, z)$, we showed that the resulting optical depth was consistent with the observational data at a reasonable confidence level. In addition, we investigated several issues which could possibly influence the optical depth, among them: the uncertainty of the solar Galacto-centric distance R_\odot , the problem of a precise determination of the rotation curve, the difference between a single exponential vertical profile and a double exponential profile, and the structure of the central bulge. We pointed out that the optical depth uncertainty connected with each of these factors was relatively small, and could not spoil the consistency between the model predictions and the observations.

The purpose of the present work is to check the microlensing optical depth predictions within a finite-width disk model framework, assuming a spatial mass distribution derived directly from the interpolating rotation curve, as it was described in Sect. 2.1. In this model the disk thickness is crucial and it affects the distribution of column mass density which can change while keeping the predicted rotation unchanged and identical with the interpolating rotation curve of Sect. 2.1. As it was mentioned earlier, we assume a volume mass density of the form $\rho(r, z) = \sigma(r)f(z)$, where $\sigma(r)$ is the column mass density and $f(z)$ is either the normalized “mexican hat” or exponential vertical profiles given in Eq. 1. Each of these profiles is defined by its own characteristic width-scale, h or h' , which are free parameters. Their optimal values can be constrained with the help of microlensing measurements, as it is done in the next subsection.

3.4. Microlensing in a finite-width disk model

The sources of light observed in the microlensing events are randomly distributed in the vicinity of the Galactic center. This requires some averaging in the latitudinal angle b . Hence, the optical depth observable (as a function of b) should be understood as a moving average. It is frequently assumed that the sources are located on the symmetry axis. We adopt this simplification in calculating $\tau(b)$ by substituting $l = 0$ and $\chi = 0$ in Eq. 7.

Our idea of determining $\tau(b)$ is simple. For a fixed width-scale h or h' , we obtain a volume mass density $\rho(r, z)$ from the Galactic rotation curve, as described in detail in Sect. 2.1. With this $\rho(r, z)$ we calculate the optical depth $\tau(b)$ from Eq. 7, substituting $l = 0$ and $\chi = 0$. The results for the vertical density

profiles Eq. 1 are shown in Fig. 2 and Fig. 3, respectively. The

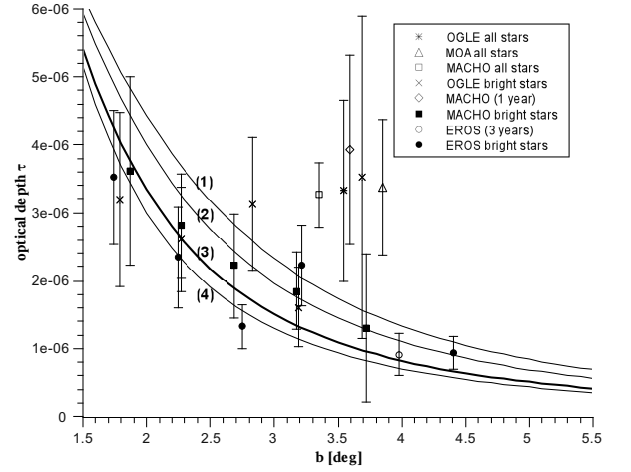


Fig. 2: [SOLID LINES]: the optical depth $\tau(b)$ in the finite-width disk model for the “mexican hat” vertical profile, with the width-scales: (1) $h = 180$ pc, (2) $h = 150$ pc, (3) $h = 117$ pc [the best fit], and (4) $h = 100$ pc. The points with error bars represent the measurement data collected by several collaborations (references in the text).

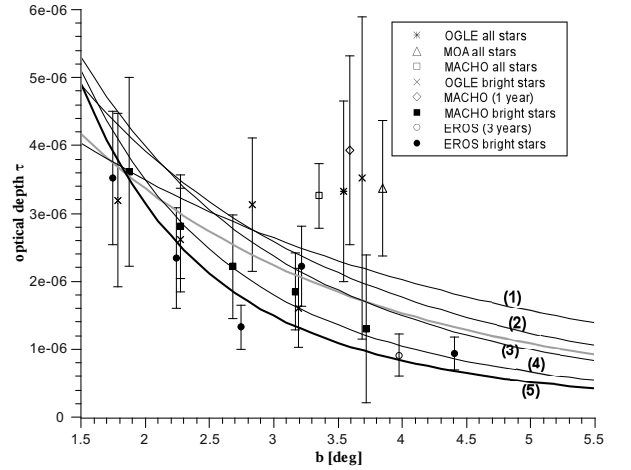


Fig. 3: [SOLID LINES]: the optical depth $\tau(b)$ in the finite-width disk model for the exponential vertical profile with the width-scales: (1) $h' = 325$ pc, (2) $h' = 200$ pc, (3) $h' = 150$ pc, (4) $h' = 100$ pc, (5) $h' = 88$ pc [the best fit]. For comparison, the [GRAY THICK LINE] represents the $\tau(b)$ in the infinitesimally thin disk model for which the equivalent volume mass density was obtained assuming the exponential vertical profile with $h' = 325$ pc. The points with error bars represent measurements data collected by several collaborations (references in the text).

curves, corresponding to different width-scale values, are shown together with the observed data points described in Sect. 3.2. We remind that we are interested in the bright stars sub-sample only.

To measure the accuracy of the fitting curves obtained for various width-scales, we calculate the reduced “chi squared”, χ_{red}^2 , that is, the “chi squared” divided by the number of the degrees of freedom (the fit residuals from the bright stars sub-sample were taken into account). The χ_{red}^2 are plotted in Fig. 4 for several width-scales h (black dots in the top panel) and h' (black dots in the bottom panel). By cubic-spline-interpolating the χ_{red}^2 , we may regard it as a smooth function of the width-scale. This allows us to determine the optimum width-scale at the minimum of each χ_{red}^2 . The width-scales obtained this way

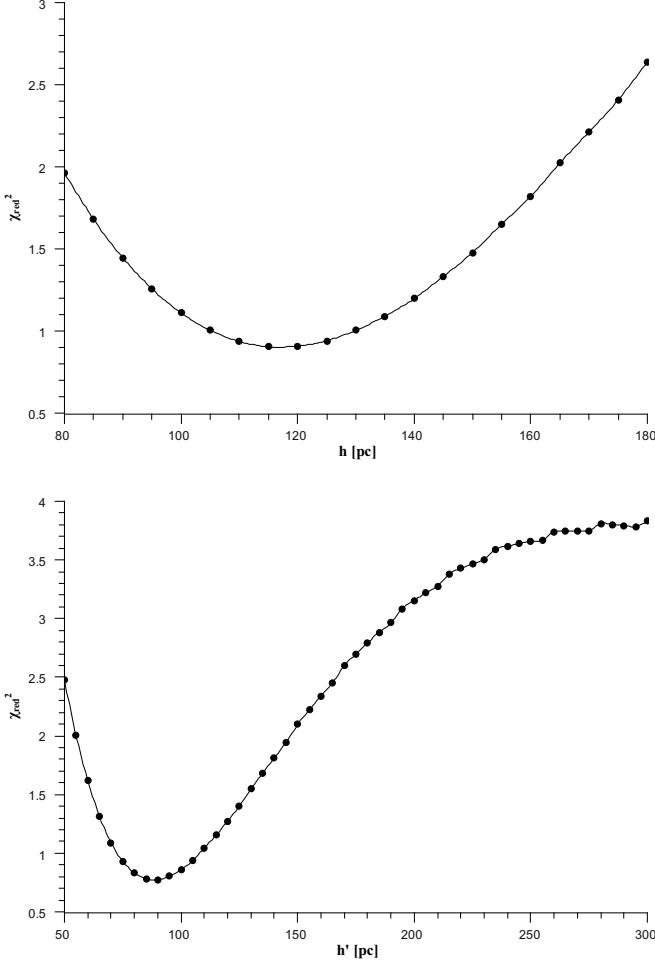


Fig. 4: The reduced “chi squared”, χ_{red}^2 , as a function of the width-scale parameter: h (for the “mexican hat” vertical profile [TOP PANEL]) or h' (for the exponential vertical profile [BOTTOM PANEL]). The values of the width-scales that minimize the χ_{red}^2 are: $h = 117$ pc and $h' = 88$ pc, respectively.

are $h = 117$ pc and $h' = 88$ pc, respectively, for the “mexican hat” and the exponential vertical profiles.

Finally, we need to verify whether the approximation of sources aligned to the symmetry axis is justified. To this end we performed a Monte Carlo simulation. We chose randomly a number of $n = 10^4$ pairs (l, χ) in the range $\chi \in (-0.125, 0.125)$ (corresponding to ± 1 kpc) and $l \in (-5^\circ, 5^\circ)$ for each fixed latitude b , with the probability weight directly proportional to $\rho(r, z)$, simultaneously finding the corresponding $\tau(l, b, \chi)$ as defined in Eq. 7. The resulting mean optical depth and its standard deviation are shown as functions of b in Fig. 5. The mean value is close to $\tau(0, b, 0)$ (solid line), which proves the approximation to be quite accurate.

4. The vertical gradient of rotation

In Jałocha et al. (2010) we modelled the vertical gradient of MW’s rotation in the framework of infinitesimally thin disk model. We compared the predicted high absolute gradient values with the gradient measurements by Levine et al. (2008), and found them to be consistent with each other. In what follows, to see the influence of the vertical structure of mass distribution on the gradient value and its behavior, we will repeat these studies in a more accurate model accounting for a finite disk thickness.

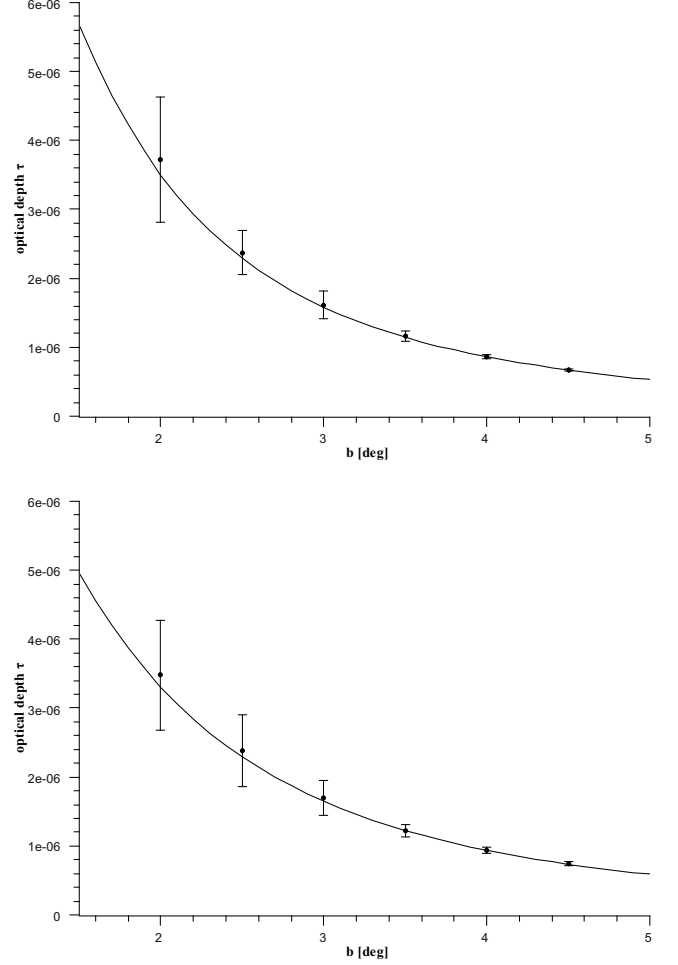


Fig. 5: The results of a Monte Carlo simulation: the mean optical depth and its standard deviation for a sample of sources chosen randomly with a probability distribution proportional to $\rho(r, z)$. [POINTS+ERROR BARS]. To compare with, the [SOLID LINE] shows the optical depth calculated with the help of the integral Eq. 7 with $l = 0$ and $\chi = 0$. The [TOP PANEL] shows the results for the “mexican hat” vertical profile with $h = 117$ pc, and the [BOTTOM PANEL] those for the exponential vertical profile with $h' = 88$ pc.

The microlensing results of Sect. 3.4 imply a width-scale of 117 pc for the disk with the “mexican hat” vertical profile, and 88 pc for that with the exponential vertical profile. With the corresponding mass distributions substituted in Eq. 3, we find our prediction for the vertical gradient of rotation in the rectangular measurement region of Levine et al. (2008): $r \in (3, 8)$ kpc and $z \in (0, 1.8)$. Our predictions for the gradient in this region are shown in Fig. 10, where they were also compared with the predictions of the infinitesimally thin disk model and with those of finite-width disks with the exponential vertical profile.

For all finite-width disks, the gradient falls off to 0 at $z = 0$, owing to the smoothness and z -reflection symmetry of the mass distribution (then, the gradient is z -antisymmetric), while for the infinitesimally thin disk with a mass distribution singular on the symmetry plane, the gradient attains a finite and large value at $z = 0$ (this requires the gradient line to be discontinuous at $z = 0$). For low altitudes, the gradient behavior is dependent on the particular structure of mass distribution decided by the parameter h which introduces a characteristic length that scales: the altitudinal extent of the “turn-overs” (seen in the gradient lines with $h > 0$), their local minimum positions, and the degree of their curvatures. Another feature evident from Fig. 10, is the

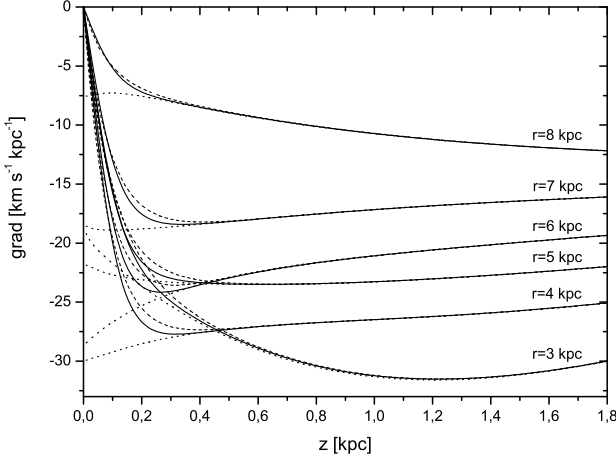


Fig. 6: The vertical gradient of rotation in finite-width disk models, as a function of altitude above the mid-plane $z = 0$, shown at various radii: [SOLID LINES] – the gradient for a disk with the “mexican hat” vertical profile ($h = 117$ pc), and [DASHED LINES] – the gradient for a disk with the exponential vertical profile ($h' = 88$ pc). To emphasize the universal gradient behavior at higher altitudes for various disks of the same mass, the results for an infinitesimally thin disk model (with the surface mass density identified with the column mass density of the disk with the “mexican hat” vertical profile) was shown – [DOTTED LINES].

fact that there is little difference between disk models with the exponential and with the “mexican hat” vertical profile.

Fig. 7 shows the behavior of the gradient as a function of z at $r = 4$ kpc for the “mexican hat” profile with various width-scales h . The smaller is h the higher is the gradient’s absolute value, but already for $|z| > 0.4$ kpc the gradient is almost independent of h . For all finite-width disk, the smaller is the width-

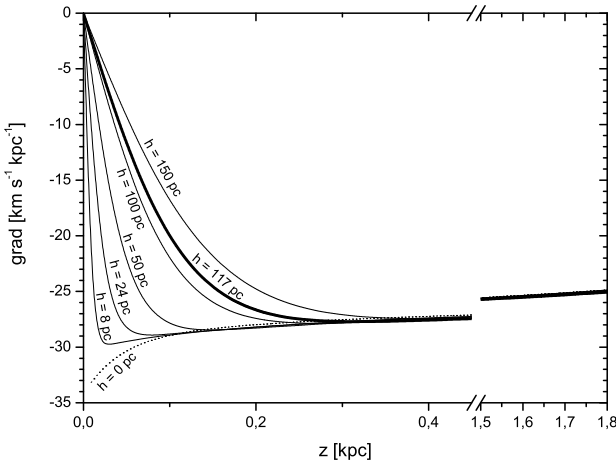


Fig. 7: The vertical gradient of rotation in a finite-width disk model with the “mexican hat” vertical profile [SOLID LINES], as a function of the altitude above the mid-plane ($z = 0$), shown at $r = 4$ kpc for various width-scales h . To compare with, the [DOTTED LINE] shows the gradient behavior in the limit of the infinitesimally thin disk model ($h \rightarrow 0$). The convergence to that limit is point-wise continuous, although not uniform – the gradient for $h = 0$ is discontinuous at $z = 0$. For higher altitudes, beyond the main concentration of masses, the gradient behavior is universal.

scale the lower is the altitude above the mid-plane at which the high gradient values characteristic of the infinitesimally disk are obtained. With increasing and large enough z , the gradient in finite-width disks gradually overlaps with that of the infinitesimally thin disk. For sufficiently large $|z|$, the differences between the predictions for disks with various h cease to be visible

and a universal asymptotics can be seen. (Note, that the total disk mass depends on h , which explains tiny differences in the asymptotics, which could be formally eliminated by rescaling masses of all disks to the same value.) Physically, this behavior is clear: for altitudes high enough, beyond the main concentration of mass in the mid-plane vicinity, the gravitational field of the infinitesimally thin disk of comparable mass, perfectly approximates that of a finite-width disk. Mathematically, this behavior becomes clear by examining the asymptotics of the integral Eq. 3, as shown in the Appendix.

In comprehending the behavior of the gradient lines, it is useful to remember, that the infinitesimally thin disk can be considered as a limit $h \rightarrow 0$ of finite-width disks of various vertical profiles. This property is readily seen in figure Fig. 7. The gradient lines of finite-width disks converge point-wise to the gradient lines of the infinitesimally thin disk, although this convergence is not uniform. Owing to this fact, the gradient lines of finite-width disks are globally continuous, but there is a discontinuity at $z = 0$ in the gradient lines for the infinitesimally thin disk. From this convergence and the universal asymptotics referred to above, one can also infer the presence of the turn overs of the gradient lines for finite-width disks (the reasoning is given in the Appendix).

Fig. 8 shows the gradient as a function of radius at fixed z for the “mexican hat” profile with various width-scales h , and additionally, for the exponential vertical profile ($h' = 88$ pc). Similarly as before, the thinner is a disk, the higher are the the gradient absolute values. At the same time the difference between the “mexican hat” vertical profile and the exponential one is negligible. From this figure one can also see that the radial variations of the vertical gradients reflect the behavior of the radial gradients of the rotation curve.

4.1. Comparison of the vertical gradient predictions with the observations

The measurements of the vertical gradient in the rotation of our Galaxy within the radial distance 3 – 8 kpc and for altitudes above the mid-plane out to 100 pc, give high gradient absolute values of $22 \pm 6 \frac{\text{km}}{\text{s} \cdot \text{kpc}}$ (Levine et al., 2008). For the rotation curve used, such high gradients may still be obtained in more realistic disk models with finite thickness, provided that the width-scales of the disks are sufficiently small. Fig. 7 shows that with the width-scale of $h = 117$ pc for the “mexican hat” vertical profile the gradient absolute value may exceed $20 \frac{\text{km}}{\text{s} \cdot \text{kpc}}$. This width-scale is favored by the analysis of the gravitational microlensing of Sect. 3.4 for the same rotation curve, assuming $\Omega_0 = 200 \frac{\text{km}}{\text{s}}$ for the circular speed of the standard of rest at the Sun position. If this rotation curve, in conjunction with the finite-width disk model, correctly describe the Galactic dynamics, then the Galactic disk has a small thickness, as suggested by our analysis. However, an increase by 10% in the rotation velocity due to the uncertain value of the speed parameter Ω_0 , might result in a similar increase in the absolute gradient value, which follows from the scaling of velocities by a (possibly r -dependent) factor.

It should be noted that some authors obtain high Ω_0 , e.g., $239 \frac{\text{km}}{\text{s}}$ (Bovy et al., 2009) or even higher, as suggested by other studies. Furthermore, a change in Ω_0 may also result in a change in the predicted width-scales of the disk models: for a higher Ω_0 we expect a corresponding increase to occur in the width-scale. E.g., by rescaling the gradients lines in Fig. 7 by a factor > 1 , a fixed gradient value at a fixed altitude will be attained on a gradient line corresponding to a higher h . Similarly, if the absolute

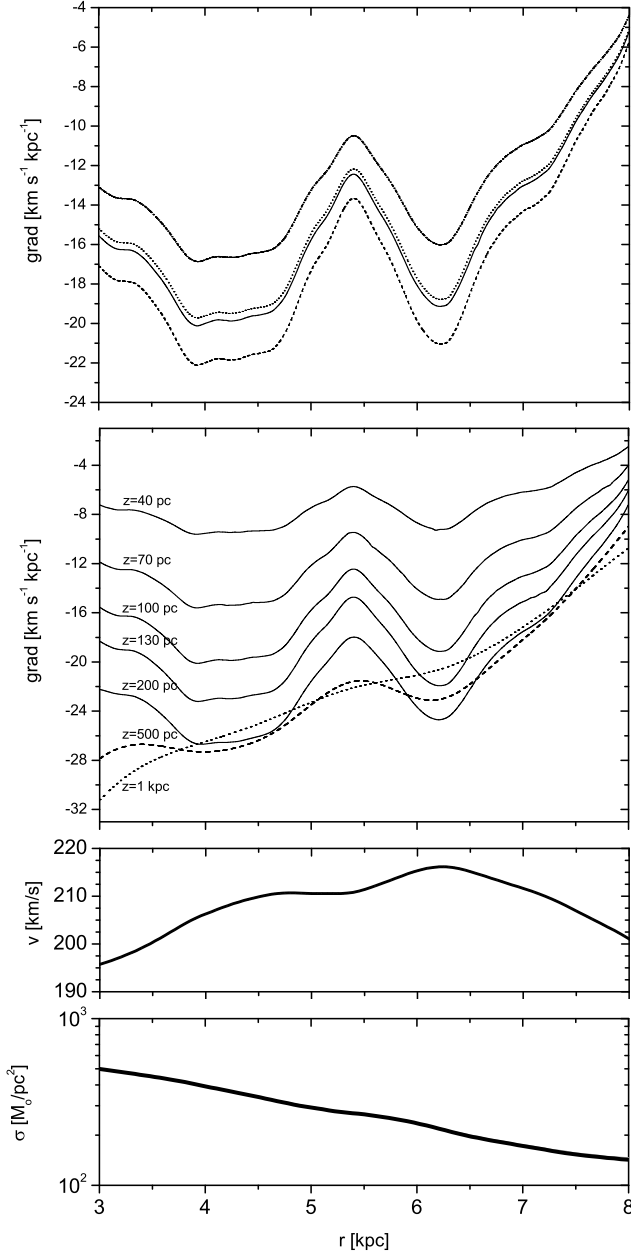


Fig. 8: The vertical gradient of rotation of MW in a finite-width disk model with “mexican hat” vertical profile as a function of the radial variable, shown for various width-scales h and altitudes z above the mid-plane. [**Top panel**]: the gradient at an altitude of 100 pc; [DASHED LINE] – $h = 100$ pc, [SOLID LINE] – $h = 117$ pc, and the uppermost [DASH-DOTTED LINE] – $h = 150$ pc. (To compare with, the [DASH-DOTTED LINE] in the middle shows the gradient in a disk model with the vertical exponential profile and the width-scale $h' = 88$ pc at the same altitude.) [**Middle Top panel**]: the vertical gradient of rotation at various altitudes for $h = 117$ pc. To enable the comparison of structures in the vertical gradient with those in the rotation curve and in the column mass density, [**Middle Bottom panel**] shows the MW rotation curve, and [**Bottom panel**] shows the column mass density for a finite-width disk model with “mexican hat” vertical profile ($h = 117$ pc).

gradient value reaches the higher end of the range from $16 \frac{\text{km}}{\text{s} \cdot \text{kpc}}$ to $28 \frac{\text{km}}{\text{s} \cdot \text{kpc}}$, allowable by the Levine et al. (2008) measurements, the predicted width-scales should become accordingly larger.

5. Column mass density at the Sun vicinity

5.1. The rotation curve in axial symmetry

We can regard the fragment of rotation curve (Fig. 1) inside solar radius as reliable. It is determined using a tangent point method applied to rotation data with relatively small scattering. Most importantly, no mass model was involved. Under the assumption of axial symmetry (concentric circular orbits), the tangent point method locates osculating points at extrema in the Doppler image along lines of constant galactic longitude (the method distinguishes the observer at the Sun position). The uncertainty in the resulting rotation curve lies mainly in the free parameters R_0 and Ω_0 which must be taken from elsewhere. For consistency with the assumptions of the method, the resulting rotation curve must be modelled under axial symmetry (non-axisymmetric features in the rotation or in the derived quantities, such as the predicted column mass density, are outside the scope of this method).

Contrary to the situation inside solar circle, the rotation curve outside solar circle is not reliable, the rotation measurement data are characterized by large scattering and are interpreted within an assumed mass model on which the resulting external rotation curve is largely dependent.

5.2. Comparison of the local measurement with a prediction of axisymmetric model

The locally measured column mass density from star counts in the solar vicinity (a calm an empty region) is unlikely to be representative for the entire solar circle. The real column density is not axisymmetric, mostly due to the spiral structure. This factor should be taken into account when comparing a local measurement with a model prediction. In particular, the locally measured value of the column mass density does not have to agree with that predicted at the Sun position in the framework of axisymmetric disk model. This only partly explains the discrepancy between the locally determined value at the Sun vicinity and that inferred from the rotation curve in the axisymmetric disk model.

The locally determined value of $\approx 71 \pm M_\odot \text{pc}^{-2}$ (all gravitating matter below $|z| < 1.1$ kpc) was inferred from solving the vertical Jeans equation for a stellar tracer population (Kuijken & Gilmore, 1991). In a recent paper (Zhang et al., 2013) a similar analysis implies that the total gravitating column mass density is $\approx 67 M_\odot \text{pc}^{-2}$ ($|z| < 1.0$ kpc), of which the contribution from all stars is $\approx 42 M_\odot \text{pc}^{-2}$ and that from cold gas is $13 M_\odot \text{pc}^{-2}$. The value of $\approx 140 M_\odot \text{pc}^{-2}$ inferred in disk model in this paper should be compared with the local value $\approx 70 M_\odot \text{pc}^{-2}$, since the disk model describes total dynamical mass accounting for the rotation curve.

Another contribution to this discrepancy, which seems more important, may point to problems with the rotation curve outside the solar circle, as we illustrate below with the help of a toy model.

A toy model. For a flattened mass distribution, the relation between the rotation curve and the resulting column mass density is nonlocal. The local density value is strongly dependent on the local radial gradient in the rotation curve, and it also depends on the external part of rotation curve. On larger scales this relation is less important and the amount of mass counts more, like in a spherical model. In consequence of this, the local value of the surface density at the solar circle will depend on the accuracy of determining the rotation curve outside solar circle.

This influence is illustrated in Fig. 9. To obtain the part of the toy model rotation curve outside solar circle, we moving-

averaged the scattered data ($r, \sqrt{\sum_i^N \frac{r_i v_{ci}^2}{Nr}}$) for $20 \text{ kpc} > r > 7 \text{ kpc}$ (open circles in Fig. 9), taking the $r > 8 \text{ kpc}$ part only. Inside the solar circle, the rotation curve was left unchanged as it is well determined. We joined the internal and external part so as to satisfy the constraint 200 km/sec at 8 kpc . For such obtained rotation curve, the density at the solar position $r = 8 \text{ kpc}$ reduces to a value $\approx 70 \text{ M}_\odot \text{pc}^{-2}$.

We see from this example, that the problem of discrepancy in the density lies mainly in the rotation curve close to and outside the solar circle. It follows, that in preparing a better rotation curve in the future, it may be necessary taking into account constraints from the local mass density. Clearly, with the original rotation curve the density is greater. For a less flattened curve, the local density could be made to agree with the low value of about $70 \text{ M}_\odot \text{pc}^{-2}$ at 8 kpc , consistently with results of (Kuijken & Gilmore, 1991) and (Zhang et al., 2013). With the

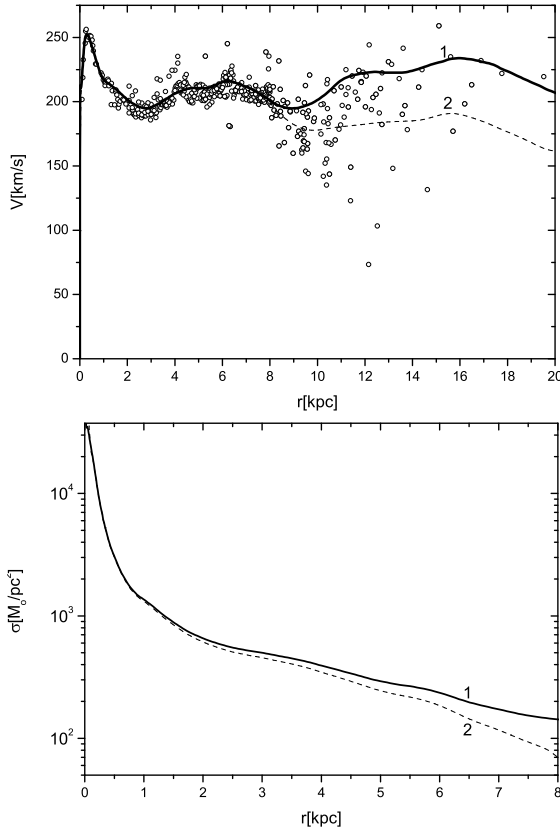


Fig. 9: Comparison of column mass densities for two rotation curves which are identical inside solar circle and differ from each other outside the circle: [Top panel] – 1[solid line]: the original rotation curve as in Fig.1, 2[dashed line]: the toy model rotation curve described in the text; [Bottom panel] – the corresponding column mass densities for a finite-width disk with the “mexican hat” vertical profile and height scale of 117 pc .

toy model curve, the results for the vertical gradient of rotation inside solar circle is not changed significantly in a large region, as shown in Fig.10, even though the predicted column density at the solar circle is significantly changed, by a factor of 2. This change would not influence much the results well inside the solar circle also for quantities which are integrals over a large region, such as the integrated optical depth we modelled in the present paper.

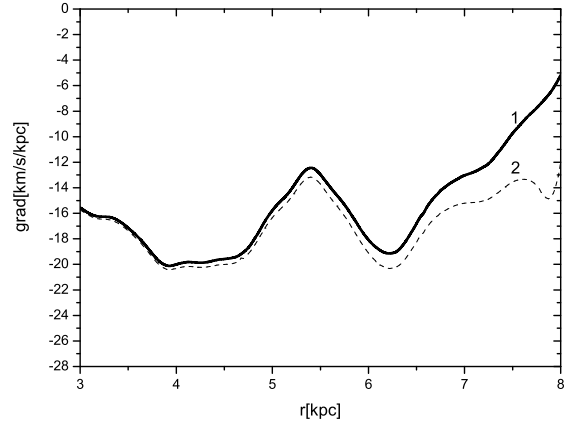


Fig. 10: The vertical gradient of rotation at $z = 100 \text{ kpc}$ for the original rotation curve and for the toy model curve. The gradient prediction is still high and not significantly changed in a region $0 - 7 \text{ kpc}$, compared to its absolute value. (The local minimum close to 8 kpc is a result of a discontinuity in the toy model curve).

6. Summary and concluding remarks

We obtained mass distribution in finite-width disks in an iterative fashion from a given rotation curve. We assumed that the volume density $\rho(r, z)$ could be factorized as $\rho(r, z) = \sigma(r)f(z)$, where $f(z)$ is a normalized vertical profile, with a characteristic width-scale which is a free parameter. We did not assume any constraints on the column mass density $\sigma(r)$, so that its functional dependence was governed entirely by the shape of the rotation curve. The density effectively describes all forms of the dynamical mass inferred from the Galactic rotation, therefore, the disk-width should not be confused with the width-scale parameters measured in the Sun vicinity for various stellar subsystems.³

Based on the gravitational microlensing measurements, it was possible within the disk model framework to determine the effective width-scales. We tested such obtained mass distributions by comparing the model predictions for the vertical gradient of the azimuthal velocity with the gradient measurements.

The (integrated) optical depth from the microlensing measurements is influenced by the amount of mass distributed along the lines of sight towards the Galactic center, whereas the details of the distribution are less important. We inferred the optimum width-scales of the considered disks by means of finding the best fits to the optical depth measured along various lines of sight. This shows that microlensing can be used as a tool to independently constrain the mass distribution models. With such determined width-scales, the resulting prediction for the behavior of the vertical gradient of rotation was compared with the gradient measurements in the mid-plane vicinity. This comparison turned out consistent with a small disk thickness. (Interestingly enough, for such determined width-scales, the effective disk widths defined by the “1/e criterion”, are almost equal: $2h' = 176 \text{ pc}$ and $\sim 1.49h = 174 \text{ pc}$, respectively, for the exponential and for the “mexican hat” vertical profiles.)

The behavior of the vertical gradient of the azimuthal velocity and its value, when calculated on low altitudes above the mid-plane, is very sensitive to the width-scale parameter. At a given altitude in the gradient measurements region, the calculated gradient value changes significantly with the width-scale

³ Incidentally, the obtained width-scale of $\approx 100 \text{ pc}$ is consistent with the distribution of massive stars in the Sun vicinity. However, massive stars are rare locally and do not contribute substantially to the mass for any proposed IMF.

parameter. When the parameter is too high the absolute gradient value is too low compared with the measurements. Higher absolute gradients at low altitudes above the mid-plane suggest a smaller effective thickness of the Galactic disk. The rotation velocity is another factor that governs the gradient value. In particular, given a disk thickness and the gradient behavior, one could constrain the allowable range for the motion of the standard of rest at the Sun position (the width-scale could be increased for higher Ω_0). Testing the vertical structure of the mass distribution with the help of the gradient measurements is thus a particularly sensitive tool, and this is therefore important to have high accuracy measurements of the gradient at small altitudes above the mid-plane.

Column mass density of flattened mass distributions is sensitive to uncertainties in the circular velocity. This sensitivity can be observed in the approximation of infinitely thin disk model (Binney & Tremaine, 1987). It is therefore important to have reliable rotation curves when studying flattened galaxies. However, Galactic rotation is relatively well known inside solar circle, therefore, with better data, we can expect some differences in the column density to occur close to or outside solar circle. These changes should not influence significantly results for global quantities inside solar circle, such as the predicted integrated optical depth or vertical gradient inside the solar circle. In particular, as shown in Sect. 5, with suitably changed external part of the rotation curve, the disk model prediction for the column density at the solar circle of $\approx 140 \text{ M}_\odot \text{ pc}^{-2}$ made with the present rotation curve, could be reduced to a local value of $\approx 70 \text{ M}_\odot \text{ pc}^{-2}$ inferred at the Sun vicinity from Jeans modelling (Kuijken & Gilmore, 1991; Zhang et al., 2013).

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Appendix A:

A.1. Derivation of Eq. 2 and Eq. 3

The kernel function J defined in the introduction is singular at an isolated point ($\zeta = 0, r = R$), and is continuous elsewhere. This singularity is integrable in the principal value sense in Eq. 2 and Eq. 3. Function J is scale-invariant. In particular, $J(r, R, \zeta) = J(r/R, 1, \zeta/R)$, which means that J is effectively dependent on two variables only: r/R and ζ/R . This property allows us to represent J on a plane as in Fig. A.1.

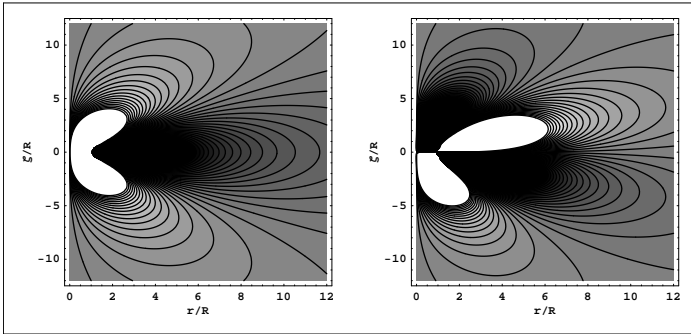


Fig. A.1: Contour plots of $\tanh(J(r, R, \zeta))$ and $\tanh(R \partial_\zeta J(r, R, \zeta))$. These functions are singular at $r = R$ and $\zeta = 0$ and continuous elsewhere.

Differentiation and taking limits can be interchanged with the integration only under particular conditions imposed on the function under the integration sign. If not said otherwise, we assume these conditions are met. For this reason, the case of infinitesimally thin disk (with $f(z) = \delta(z)$) must be treated separately.

The expression for $R \partial_R \Psi(R, Z)$ in Sect. 2 involves an integral

$$I = \int_{-\infty}^{\infty} dz \rho(r, z) J(r, R, z - Z).$$

The integral can be written as a sum $I = \int_{-\infty}^0 dz \rho(r, z) J(r, R, z - Z) + \int_0^{\infty} dz \rho(r, z) J(r, R, z - Z)$ if the two summands exist. By substituting $z \rightarrow -\tilde{z}$, the first integral can be rewritten as $\int_0^{\infty} (-d\tilde{z}) \rho(r, -\tilde{z}) J(r, R, -[\tilde{z} + Z])$, and next, since $\rho(r, -\zeta) = \rho(r, \zeta)$ and $J(r, R, -\zeta) = J(r, R, \zeta)$, as $\int_0^{\infty} d\tilde{z} \rho(r, \tilde{z}) J(r, R, \tilde{z} + Z)$. Finally, by renaming $\tilde{z} \rightarrow z$, we obtain

$$I = \int_0^{\infty} dz \rho(r, z) \cdot [J(r, R, z + Z) + J(r, R, z - Z)].$$

This proves Eq. 2.

The integral expression for the vertical gradient of rotation in the quasi-circular orbits approximation can be proved by performing a partial differentiation of I under the integration sign:

$$\partial_Z I = \int_0^{\infty} dz \rho(r, z) \cdot [\partial_Z J(r, R, z + Z) + \partial_Z J(r, R, z - Z)].$$

Now, $\partial_Z J(r, R, z \pm Z) = \pm \partial_z J(r, R, z \pm Z)$, which implies that $\partial_Z I = \int_0^{\infty} dz \rho(r, z) \cdot [\partial_z J(r, R, z + Z) - \partial_z J(r, R, z - Z)]$. Integration by parts under usual conditions gives $\partial_Z I = \mathcal{B} - \int_0^{\infty} dz \partial_z \rho(r, z) \cdot [J(r, R, z + Z) - J(r, R, z - Z)]$, where $\mathcal{B} = \left(\lim_{z \rightarrow +\infty} \rho(r, z) [J(r, R, z + Z) - J(r, R, z - Z)] \right) - (\rho(r, 0^+) [J(r, R, Z) - J(r, R, -Z)])$ is a boundary term. The expression in the first round bracket in this term is zero for any finite Z, r , on account of the vanishing of the difference in the square bracket and the vanishing of ρ in the same limit (it suffices to assume a finite support of ρ). The expression in the second round bracket also vanishes for finite $\rho(r, 0)$ since J is an even function of the third argument. Hence, the boundary term \mathcal{B} also vanishes. Formally, one should also verify whether the integration with respect to r and taking the above limit $z \rightarrow +\infty$ commute, as there is an integration over r present in the expression for $R \partial_R \Psi$. When this requirement is met, we have

$$\partial_Z \int_0^{\infty} dr I = \int_0^{\infty} dr \int_0^{\infty} dz \partial_z \rho(r, z) \cdot [J(r, R, z - Z) - J(r, R, z + Z)],$$

which proves Eq. 3.

A.2. A special case: the exponential vertical profile

In the case of the vertical exponential falloff of the density profile, $\rho(r, z) = \frac{e^{-|z|/h}}{2h} \sigma(r)$, calculation of integrals Eq. 2 and Eq. 3 can be simplified. Then, $v_\phi^2(R, Z) = \frac{G}{v_\phi(R, Z)} \int_0^\infty \int_0^\infty dz \sigma(r) \frac{e^{-z/h}}{2h} \cdot [J(r, R, z - Z) + J(r, R, z + Z)]$ and $\partial_Z v_\phi(R, Z) = \frac{G}{v_\phi(R, Z)} \int_0^\infty \int_0^\infty dz \sigma(r) \frac{e^{-z/h}}{2h^2} \cdot [J(r, R, z - Z) - J(r, R, z + Z)]$. Now, both $v_\phi(R, Z)$ and $\partial_Z v_\phi(R, Z)$ can be expressed in terms of two integrals I_+ and I_- :

$$I_\pm = \frac{G}{h} \int_0^\infty \sigma(r) dr \int_0^\infty dz e^{-z/h} J(r, R, z \pm Z).$$

Namely, for $\rho(r, z) = \frac{e^{-|z|/h}}{2h} \sigma(r)$, $v_\phi(R, Z) = \sqrt{I_+ + I_-}$ and $\partial_Z v_\phi(R, Z) = \frac{I_+ - I_-}{2h \sqrt{I_+ + I_-}}$.

A.3. Qualitative properties of the vertical gradient of rotation and the presence of “turn-overs”.

Eq. 3 involves an integral $\int_0^\infty dz f'(z) [J(r, R, z - Z) - J(r, R, z + Z)]$

which equals $\mathcal{K} = \int_{-\infty}^\infty dz f'(z) J(r, R, Z - z)$, owing to the symmetry of J and f . Now, consider $|Z| > A > 0$ large enough, beyond the main mass concentration, such that $\int_{-A}^A f(z) dz \approx 1$

(by assumption $\int_{-\infty}^\infty f(z) dz = 1$). For such Z , only a region $|z| \ll |Z|$ contributes to \mathcal{K} , and we can use the approximation formula $J(r, R, Z - z) - J(r, R, Z) \approx -z \partial_Z J(r, R, Z)$ to obtain

$\mathcal{K} \approx J(r, R, Z) \int_{-\infty}^\infty f'(z) dz - \partial_Z J(r, R, Z) \int_{-\infty}^\infty z f'(z) dz$. Since $f(z)$ and $z f(z)$ vanish at the infinity, integrating by parts gives

$\mathcal{K} \approx \partial_Z J(r, R, Z) \int_{-\infty}^\infty f(z) dz = \partial_Z J(r, R, Z)$. On the other hand,

for $f(z) = \delta(z)$ and for $Z \neq 0$, $\mathcal{K} = \int_{-\infty}^\infty dz \delta'(z) J(r, R, Z - z) =$

$-\int_{-\infty}^\infty dz \delta(z) \partial_Z J(r, R, Z - z) = \partial_Z J(r, R, Z)$. Hence, we obtain an

intuitively clear result: *for all finite-width thin disks with the same column mass density $\sigma(r)$, the behavior of the vertical gradient at altitudes large enough, is universal and the same as for an infinitesimally thin disk with surface mass density $\sigma(r)$.*

Another qualitative result is obtained in the limit $Z \rightarrow 0$. For $Z \neq 0$, $\mathcal{K}(R, r, -Z) = -\mathcal{K}(R, r, Z)$, thus $\lim_{Z \rightarrow 0} \mathcal{K} = 0$ by continuity of \mathcal{K} as a function of Z , and the vertical gradient is zero at $Z = 0$, at least for those mass distributions for which the usual theorems on the continuity of functions defined by integrals Eq. 3 apply.

However, there is an exception from the above continuity behavior of the gradient lines at $Z = 0$. It is important to remember that the operation of taking various limits and the operation of integration are not interchangeable in general. In particular, an integral of a function sequence consisting of continuous functions with a parameter can result in a discontinuous function of that parameter. For $f(z) = \delta_n(z)$, where δ_n is a functional sequence representing the Dirac δ , the result of continuity of the gradient

does not necessarily follow and we can have a nonzero value in the same limit, in which case the integral Eq. 3 is discontinuous at $Z = 0$. To give a simple example of what then may happen, consider a function sequence $u_n(x, y) = \frac{2xy^2}{\pi(1+x^2)(\frac{1}{n^2}+x^2+y^2)^2}$, then $\int_{-\infty}^{+\infty} u_n(x, y) dy = g_n(x)$, with $g_n(x) = \frac{nx}{(1+x^2)\sqrt{1+n^2x^2}}$. Now, consider a limiting function $g(x) = \lim_{n \rightarrow +\infty} g_n(x)$ and see if it is continuous at $x = 0$. For $x = 0$, $g(0) \equiv \lim_{n \rightarrow +\infty} g_n(0) = 0$, whereas for $x \neq 0$, $g(x) \equiv \lim_{n \rightarrow +\infty} g_n(x) = \frac{1}{1+x^2} \neq 0$, thus, $g(0) = 0 \neq 1 = \lim_{x \rightarrow 0} g(x)$, therefore $g(x)$ is discontinuous at $x = 0$. Now, think of the gradient lines in Fig. 7 – then the finite disk corresponds to the situation described by $g_n(x)$ ($h = n^{-1} > 0$), whereas the infinitesimally thin disk corresponds to the situation of discontinuous $g(x)$ ($h = 0$).

Finally, we may try to understand the occurrence of the “turn-overs” in the gradient lines for $h > 0$, such as those seen in Fig. 10 or Fig. 7. First, note that a gradient line must asymptotically converge to zero, which is the universal asymptotics property discussed earlier. Second, the gradient line starts from 0 at $Z = 0$, which we have also seen above. Now, let’s perform a mapping of the region $0 < Z < +\infty$ to an interval $0 < Z < 1$ by means of a transformation $Z \rightarrow \tanh Z$. Then the transformed gradient lines are continuous for $0 \leq Z \leq 1$ and vanish at the boundaries. Next, we apply the Rolle’s theorem on continuously differentiable functions vanishing on the boundaries of a compact and simply connected interval, and we infer that there must be at least one point inside the interval where the gradient line has a local minimum, which explains the presence of a “turn-over”. The Rolle’s theorem does not apply to gradient lines of the infinitesimally thin disk, because of the discontinuity, and the analogous “turn-overs” do not have to occur, which is the case in Fig. 10 or Fig. 7.

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